International Journal of Theoretical Physics, Vol. 16, No. 6 (1977), pp. 419–424

On the Rescaling Problem of Space-Times Admitting Groups of Conformal Motions

B. LUKÁCS and Á. SEBESTYÉN

Central Research Institute for Physics, H-1525 Budapest 114, P.O. Box 49, Hungary

Received: 25 *January* 1977

Abstract

The necessary and sufficient conditions are given that a space-time admitting a group of conformal motions can be mapped conformally on a space-time admitting the same group but of Killing symmetries.

1. *Introduction*

The possible relevance of space-times admitting conformal Killing symmetries has been emphasized concerning the large-scale structure of the universe (see, e.g., Geroch, 1969; Hawking and Ellis, 1973; Katzin et al., 1969), twistor theory (Dighton, 1975), and solutions of Einstein's equations with matter (Singh and Abdussattar, 1974). However, the group structure of conformal Killing fields has not been investigated in every respect, although in some cases it is supposed that the space-time admits a group of conformal motions. The present paper deals with the rescaling problem of Riemannian manifolds admitting such groups.

If a contravariant vector field K^a of a simply connected Riemannian manifold V_n satisfies the conformal Killing equation[1]

$$K_{a;b} + K_{b;a} + kg_{ab} = 0 \tag{1.1}$$

then, adopting a coordinate system such that $K^a = \delta_1{}^a$, (1.1) leads to

$$g_{ab,1} + kg_{ab} = 0$$

which can be integrated to give

$$g_{ab} = \exp\left[-\int k\, dx^1\right] g_{ab}^{(0)}$$

where $g_{ab,1}^{(0)} = 0$; thus the g_{ab} can be conformally rescaled to yield a $g_{ab}^{(0)}$ for which K^a is a Killing vector. Conversely, if K^a is a Killing vector for some $g_{ab}^{(0)}$

$$g_{ab,r}^{(0)} K^r + g_{ar}^{(0)} K^r{}_{,b} + g_{br}^{(0)} K^r{}_{,a} = 0$$

[1] The comma and the semicolon stand for partial and covariant derivatives, respectively.

then for a g_{ab} conformal to $g_{ab}^{(0)}$

$$g_{ab} = g_{ab}^{(0)} e^{-\varphi}$$

we regain equation (1.1) with $k = K^r \varphi_{,r}$.

If, however, V_n admits a group G_κ of conformal motions with generators $K_\alpha{}^a$ ($\alpha = 1, 2, \ldots, \kappa$), then in general one cannot rescale g_{ab} in such a way that for the resulting metric tensor all the vectors K_α^a be Killing fields.

In the following section we prove a theorem showing that for simply transitive groups such rescaling is always possible, whereas the theorem proved in the third part gives the necessary and sufficient conditions for the existence of such a conformal transform in the case of nonsimply transitive groups. In the fourth section some corollories of these theorems are investigated.

2. *Simply Transitive Groups*

We consider the case when V_n admits a simply transitive group G_κ of conformal motions with generators K_α^a. We have

$$K_{\alpha a;b} + K_{\alpha b;a} + k_\alpha g_{ab} = 0 \tag{2.1}$$

$$[K_\alpha, K_\beta]_a \equiv K_\alpha{}^r K_{\beta a;r} - K_\beta{}^r K_{\alpha a;r} = C_{\alpha\beta\rho} K_{\rho a}{}^2 \tag{2.2}$$

$$C_{\alpha\beta\rho} = -C_{\beta\alpha\rho} \tag{2.3}$$

($\alpha, \beta, \rho = 1, 2, \ldots, \kappa$) the $C_{\alpha\beta\gamma}$'s being the structure constants of G_κ. Simple transitivity means that the rank of the matrix formed by the K_α^a's is κ. As a consequence of the integrability conditions of (2.1) we have (Eisenhart, 1966)

$$K_{\alpha a;bc} = R_{abcr} K_\alpha^r + \tfrac{1}{2}(k_{\alpha,a} g_{bc} - k_{\alpha,b} g_{ac} - k_{\alpha,c} g_{ab}) \tag{2.4}$$

Taking the covariant derivative of (2.2) with respect to X^b, symmetrizing in a and b, and making use of (2.1) and (2.4), we get

$$K_\alpha^r k_{\beta,r} - K_\beta^r k_{\alpha,r} = C_{\alpha\beta\rho} k_\rho \tag{2.5}$$

Now we prove the following theorem:

Theorem 1. There exists a scalar φ such that

$$k_\alpha = K_\alpha^r \varphi_{,r} \qquad (\alpha = 1, 2, \ldots, \kappa)$$

Proof. Define

$$L_\alpha^a = K_\alpha^a \qquad (a = 1, 2, \ldots, n)$$

$$L_\alpha^{n+1} = -k_\alpha$$

Introducing an additional variable X^{n+1} for which $K_\alpha{}^a{}_{,n+1} = k_{\alpha,n+1} = 0$ it is seen that equations (2.2) and (2.5) can be summarized

$$L_\alpha^{\bar r} L_\beta^{\bar a}{}_{,\bar r} - L_\beta^{\bar r} L_\alpha{}^{\bar a}{}_{,r} = C_{\alpha\beta\rho} L_\rho{}^{\bar a} \qquad (\bar a, \bar r = 1, 2, \ldots, n+1) \tag{2.6}$$

2 There is a summation for Greek indices occurring twice in an expression.

These are the conditions that

$$L_\alpha^{\bar{r}}\Psi_{,\bar{r}} \equiv K_\alpha{}^r\Psi_{,r} - k_\alpha\Psi_{,n+1} = 0 \tag{2.7}$$

be a complete Jacobian system. In consequence of the simple transitivity of G_κ the number of the independent solutions of (2.7) is $n + 1 - \kappa$, whereas that of the solutions of

$$K_\alpha{}^r\Phi_{,r} = 0 \tag{2.8}$$

is $n - \kappa$. Since any Φ satisfying (2.8) is a solution of (2.7) too, there must exist a $\Psi^{(0)}$ such that

$$L_\alpha^{\bar{r}}\Psi^{(0)}{}_{,\bar{r}} = 0 \tag{2.9}$$

and $\Psi^{(0)}{}_{,n+1} \neq 0$; otherwise the number of the independent functions satisfying (2.7) would be equal to that of the functions fulfilling (2.8). Hence any solution of (2.7) is of the form

$$\Psi = \Psi(\Phi_1, \ldots, \Phi_{n-\kappa}, \Psi^{(0)}) \tag{2.10}$$

where $\Phi_1, \ldots, \Phi_{n-\kappa}$ are the solutions of (2.8). Differentiating (2.7) with respect to X^{n+1} we have

$$L_\alpha^{\bar{r}}\Psi_{,n+1},_{\bar{r}} = 0$$

showing that if Ψ satisfies (2.7) then so does $\Psi_{,n+1}$. Hence for $\Psi^{(0)}$ in (2.9) we have, in view of (2.10),

$$\Psi^{(0)}{}_{,n+1} = \chi(\Phi_1, \ldots, \Phi_{n-\kappa}, \Psi^{(0)}) \tag{2.11}$$

Now since the function

$$\varphi = \int \frac{1}{\chi}\, d\,\Psi^{(0)} \tag{2.12}$$

is again of the form (2.10), it satisfies (2.7), and in consequence of (2.11) and (2.12) we have

$$\varphi_{,n+1} = \frac{\partial\varphi}{\partial\Psi^{(0)}}\ \Psi^{(0)}{}_{,n+1} = 1$$

Hence we have

$$L_\alpha^{\bar{r}}\varphi_{,\bar{r}} \equiv K_\alpha{}^r\varphi_{,r} - k_\alpha = 0 \tag{2.13}$$

QED.

If we define

$$g_{ab}^{(0)} = g_{ab}e^{\varphi} \tag{2.14}$$

we have in consequence of (2.1) and (2.13)

$$g_{ab,r}^{(0)}K_\alpha{}^r + g_{ar}^{(0)}K_\alpha{}^r{}_{,b} + g_{br}^{(0)}K_\alpha{}^r{}_{,a} = 0$$

showing that the rescaling (2.14) yields a $g_{ab}^{(0)}$ for which all the vectors $K_\alpha{}^a$ are Killing symmetries.

3. *Nonsimply Transitive Groups*

If G_κ of the previous section is not simply transitive, meaning that the rank of the matrix of the $K_\alpha{}^a$'s is less than κ, say $\kappa - \lambda$, then there exists a set of linearly independent functions $U_{A\rho}$ such that

$$U_{A\rho} K_{\rho a} = 0 \qquad (A = 1, 2, \ldots, \lambda < \kappa; \rho = 1, 2, \ldots, \kappa) \tag{3.1}$$

and these equations must be appended to (2.1)-(2.5). By means of, e.g., Schmidt's orthogonalization

$$U_{A\rho} U_{B\rho} = \delta_{AB} \tag{3.2}$$

can be achieved.

In consequence of (2.1) and (3.1) we have

$$U_{A\rho,b} K_{\rho a} + U_{A\rho,a} K_{\rho b} - U_{A\rho} k_\rho g_{ab} = 0$$

yielding

$$K_\rho{}^r U_{A\rho,r} = \tfrac{1}{2} n\, U_{A\rho} k_\rho \tag{3.3}$$

Using (2.2) one gets for the Lie bracket of (3.1) and $K_\alpha{}^a$

$$(K_\alpha{}^r U_{A\rho,r} + U_{A\sigma} C_{\alpha\sigma\rho}) K_{\rho a} = 0$$

which must be a consequence of (3.1); thus there exist functions $D_{\alpha AB}$ such that[3]

$$K_\alpha{}^r U_{A\beta,r} = U_{A\rho} C_{\rho\alpha\beta} + D_{\alpha AR}\, U_{R\beta} \tag{3.4}$$

According to (3.1) we have

$$U_{A\rho} U_{B\sigma} C_{\rho\sigma\alpha} + U_{B\rho} D_{\rho AR}\, U_{R\alpha} = 0$$

which in view of (2.3) and (3.2) yields

$$U_{R\rho} D_{\rho RA} = 0 \tag{3.5}$$

From (3.2) and (3.4) we also have

$$D_{\alpha AB} + D_{\alpha BA} = U_{A\rho} U_{B\sigma} (C_{\alpha\rho\sigma} + C_{\alpha\sigma\rho}) \tag{3.6}$$

According to (3.4) we get from (3.3)

$$\tfrac{1}{2} n\, U_{A\rho} k_\rho = U_{A\rho} C_{\rho\sigma\sigma} + U_{R\rho} D_{\rho AR}$$

which can be rewritten using (3.5) and (3.6)

$$U_{A\rho} k_\rho = (2/n) U_{A\rho} (\delta_{\mu\nu} - U_{R\mu} U_{R\nu}) C_{\rho\mu\nu} \tag{3.7}$$

[3] There is a summation for capital Latin indices occurring twice in an expression.

Now we prove the following theorem:

Theorem 2. The necessary and sufficient conditions that a scalar φ exist such that

$$k_\alpha = K_\alpha{}^r \varphi_{,r}$$

are that the quantities

$$U_{A\rho} k_\rho = (2/n) U_{A\rho} (\delta_{\mu\nu} - U_{R\mu} U_{R\nu}) C_{\rho\mu\nu} \qquad (A = 1, 2, \ldots, \lambda < \kappa)$$

vanish.

Proof. Suppose first that

$$U_{A\rho} k_\rho = 0 \tag{3.8}$$

Defining again $L_\alpha{}^{\bar{a}}$ ($\bar{a} = 1, 2, \ldots, n + 1$), as was done in the proof of Theorem 1 of Section 2, we get (2.6) again, these being the conditions that

$$L_\alpha{}^{\bar{r}} \Psi_{,\bar{r}} \equiv K_\alpha{}^r \Psi_{,r} - k_\alpha \Psi_{,n+1} = 0 \tag{3.9}$$

form a complete Jacobian system. However, in view of (3.1) and (3.8) now we have

$$U_{A\rho} L_\rho{}^{\bar{a}} = 0 \qquad (\bar{a} = 1, 2, \ldots, n + 1)$$

showing that the number of independent equations of (3.9) is $\kappa - \lambda$, and in consequence of (3.1) the same applies also for the system

$$K_\alpha{}^r \Phi_{,r} = 0 \tag{3.10}$$

Hence the number of the independent solutions of (3.9) is $n + 1 - (\kappa - \lambda)$, whereas that of the solutions of (3.10) is $n - (\kappa - \lambda)$. From this point we follow the proof of Theorem 1 of Section 2 to conclude that there exists a scalar φ that satisfies

$$K_\alpha{}^r \varphi_{,r} - k_\alpha = 0 \tag{3.11}$$

Conversely, in view of (3.1), equations (3.11) lead to (3.8). QED.

4. *Conclusions*

From (3.7) it is seen that if the structure constants $C_{\alpha\beta\gamma}$ are antisymmetric in the last two indices then a rescaling that reduces the conformal problem to a Killing problem is always possible, hence in consequence of Theorems 1 and 2 and (3.7) we have the following:

Corollary A. If a space-time V_4 with a metric tensor g_{ab} admits a group $SO(3)$ of conformal motions then there exists a space-time $V_4{}^{(0)}$ with $g_{ab}^{(0)}$ such that

$$g_{ab}^{(0)} = g_{ab} e^{\varphi}$$

and $V_4{}^{(0)}$ admits a group $SO(3)$ of Killing symmetries.

Proof. If the group $SO(3)$ is simply transitive then apply Theorem 1; if not then consider the structure constants of $SO(3)$:

$$C_{\alpha\beta\gamma} = -\epsilon_{\alpha\beta\gamma} \qquad (\alpha, \beta, \gamma = 1, 2, 3)$$

As these are antisymmetric in the last two indices, we have according to (3.7)

$$U_{A\rho} k_\rho = 0$$

and now Theorem 2 applies. QED.

Consider now the group $SO(4)$:

$$[K_\alpha, K_\beta]_a = -\epsilon_{\alpha\beta\rho} K_{\rho a}$$
$$[K_\alpha, K_{\beta+3}]_a = -\epsilon_{\alpha\beta\rho} K_{\rho+3a}$$
$$[K_{\alpha+3}, K_{\beta+3}]_a = -\epsilon_{\alpha\beta\rho} K_{\rho a} \qquad (\alpha, \beta, \rho = 1, 2, 3) \qquad (4.1)$$

It can easily be seen from (4.1) that the structure constants of $SO(4)$ are antisymmetric in the last two indices, hence in view of (3.7) and Theorem 2 we have the following:

Corollary B. If a space-time V_4 with g_{ab} admits a group $SO(4)$ of conformal motions—being necessarily nonsimply transitive—then there exists a space-time $V_4^{(0)}$ with $g_{ab}^{(0)}$ such that

$$g_{ab}^{(0)} = g_{ab} e^{\varphi}$$

and $V_4^{(0)}$ admits a group $SO(4)$ of Killing symmetries.

As is well known, a space-time $V_4^{(0)}$ admitting a group $SO(4)$ of Killing symmetries is of the Robertson-Walker type and its metric tensor $g_{ab}^{(0)}$ can be given the form

$$g_{00}^{(0)} = a, \qquad g_{11}^{(0)} = -a, \qquad g_{22}^{(0)} = -a \sin^2 x^1$$
$$g_{33}^{(0)} = -a \sin^2 x^1 \sin^2 x^2, \qquad g_{ik}^{(0)} = 0 \qquad (i \neq k)$$

where $a_{,i} = 0$ $(i = 1, 2, 3)$. From these it is seen that for $V_4^{(0)}$ $T^a = \delta_0{}^a$ is a conformal Killing vector. Since conformal spaces have the same set of conformal Killing vectors in consequence of Corollary B, we have the following:

Corollary C. If a space-time admits a group $SO(4)$ of conformal motions then it admits a timelike conformal motion too.

References

Dighton, K. (1975). *International Journal of Theoretical Physics*, **14**, 137.

Eisenhart, L. P. (1966). *Riemannian Geometry*, Princeton University Press, Princeton, New Jersey.

Geroch, R. (1969). *Communications in Mathematical Physics*, **13**, 180.

Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, The University Press, Cambridge.

Katzin, G. H., Levine, J., and Davis, W. R. (1969). *Journal of Mathematical Physics*, **10**, 617.

Singh, K. P. and Abdussattar (1974). *General Relativity and Gravitation*, **5**, 115.